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On the approximation exponents for subspaces of $\mathbb{R}^n$

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This paper follows the generalisation of the classical theory of Diophantine approximation to subspaces of $\mathbb{R}^n$ established by W. M. Schmidt in 1967. Let A and B be two subspaces of $\mathbb{R}^n$ of respective dimensions d and e with $d+e \leqslant n$. The proximity between A and B is measured by $t=\min(d,e)$ canonical angles $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_t \leqslant \frac{\pi}{2}$; we set $\psi_j(A,B)=\sin\theta_j$. If B is a rational subspace, its complexity is measured by its height $H(B)=\mathrm{covol}(B\cap\mathbb{Z}^n)$. We denote by $\mu_n(A|e)_j$ the exponent of approximation defined as the upper bound (possibly equal to $+\infty$) of the set of $\beta>0$ such that the inequality $\psi_j(A,B)\leqslant H(B)^{-\beta}$ holds for infinitely many rational subspaces B of dimension e. We are interested in the minimal value $\mathring{\mu}_n(d|e)_j$ taken by $\mu_n(A|e)_j$ when A ranges through the set of subspaces of dimension d of $\mathbb{R}^n$ such that for all rational subspaces B of dimension e one has $\dim(A\cap B)<j$. We show that $\mathring{\mu}_4(2|2)_1=3$, $\mathring{\mu}_5(3|2)_1\le 6$ and $\mathring{\mu}_{2d}(d|\ell)_1\le 2d^2/(2d-\ell)$. We also prove a lower bound in the general case, which implies that $\mathring{\mu}_n(d|d)_d\to 1/d$ as $n\to+\infty$.

1. Introduction

The classical theory of Diophantine approximation studies how well points of $\mathbb{R}^n$ can be approximated by rational points. Here, we are interested in a problem studied by W. M. Schmidt [1967], which consists in approximating subspaces of $\mathbb{R}^n$ by rational subspaces. The results presented here can be found in my Ph.D. thesis (see [Joseph 2021] Chapters 3 and 4 for more details).

A subspace of $\mathbb{R}^n$ is said to be *rational* whenever it admits a basis of vectors with rational coordinates. Denote by $\mathfrak{R}_n(e)$ the set of rational subspaces of dimension e of $\mathbb{R}^n$. A subspace A of $\mathbb{R}^n$ is called *(e,j)-irrational* whenever $\dim(A\cap B)<j$ for all $B\in\mathfrak{R}_n(e)$; notice that being $(e,1)$-irrational is equivalent to trivially intersecting all subspaces of $\mathfrak{R}_n(e)$. Denote by $\mathfrak{I}_n(d,e)_j$ the set of all (e,j)-irrational subspaces of dimension d of $\mathbb{R}^n$.

Let us define a notion of *complexity* for a rational subspace and a notion of *proximity* between two subspaces, which will lead to the formulation of the main problem.

Let $B\in\mathfrak{R}_n(e)$; one can choose $\Xi\in\mathbb{Z}^N$, with $N=\binom{n}{e}$, a vector with setwise coprime coordinates in the class of Plücker coordinates of B. Let us define the *height* of B to be the Euclidean norm of Ξ:

$$H(B)=\|\Xi\|.$$

Endow $\mathbb{R}^n$ with the standard Euclidean norm, and define the distance between two vectors $X,Y\in\mathbb{R}^n\setminus\{0\}$ by

$$\psi(X,Y)=\sin\widehat{(X,Y)}=\frac{\|X\wedge Y\|}{\|X\|\cdot\|Y\|},$$

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where $X \wedge Y$ is the exterior product of X and Y, and the Euclidean norm $\|\cdot\|$ is naturally extended to $\Lambda^2(\mathbb{R}^n)$ so that $\|X \wedge Y\|$ is the area of the parallelogram spanned by X and Y. Let A and B be two subspaces of $\mathbb{R}^n$ of dimensions d and e respectively. One can define by induction $t = \min(d, e)$ angles between A and B. Let us define

$$\psi_1(A, B) = \min_{\substack{X \in A\setminus\{0\} \\ Y \in B\setminus\{0\}}} \psi(X, Y)$$

and denote by X_1 and Y_1 unitary vectors such that $\psi(X_1, Y_1) = \psi_1(A, B)$. Then, by induction, it is assumed that $\psi_1(A, B), \dots, \psi_j(A, B)$ have been constructed for $j \in \{1, \dots, t-1\}$, associated with pairs of vectors $(X_1, Y_1), \dots, (X_j, Y_j) \in A \times B$ respectively. One denotes by A_j the orthogonal complement of $\operatorname{Span}(X_1, \dots, X_j)$ in A and by B_j the orthogonal complement of $\operatorname{Span}(Y_1, \dots, Y_j)$ in B. Let us define in a similar fashion

$$\psi_{j+1}(A, B) = \min_{\substack{X \in A_j\setminus\{0\} \\ Y \in B_j\setminus\{0\}}} \psi(X, Y)$$

and denote by X_{j+1} and Y_{j+1} unitary vectors such that $\psi(X_{j+1}, Y_{j+1}) = \psi_{j+1}(A, B)$.

These angles between A and B are canonical in the sense of this paragraph, based on [Schmidt 1967, Theorem 4]. This will also be used to prove Claim 6.2 in Section 6 below. There exist orthonormal bases $(X_1, \dots, X_d)$ and $(Y_1, \dots, Y_e)$ of A and B respectively, and real numbers $0 \leqslant \theta_t \leqslant \cdots \leqslant \theta_1 \leqslant 1$ such that for all $i \in \{1, \dots, d\}$ and for all $j \in \{1, \dots, e\}$, $X_i \cdot Y_j = \delta_{i,j} \cos\theta_i$, where δ is the Kronecker delta and $\cdot$ is the canonical scalar product on $\mathbb{R}^n$. Moreover, the numbers $\theta_1, \dots, \theta_t$ are independent of the bases $(X_1, \dots, X_d)$ and $(Y_1, \dots, Y_e)$ chosen. Notice that $\psi_j(A, B) = \sin\theta_j$.

We can now formulate the main problem. Let $n \geqslant 2$, $d, e \in \{1, \dots, n-1\}$ such that $d + e \leqslant n$, $j \in \{1, \dots, \min(d, e)\}$, and $A \in \mathfrak{I}_n(d, e)_j$. Let us define by $\mu_n(A\,|\,e)_j$ the upper bound (possibly equal to $+\infty$) of all $\beta > 0$ such that

$$\psi_j(A, B) \leqslant \frac{1}{H(B)^\beta}$$

holds for infinitely many $B \in \mathfrak{R}_n(e)$. One also defines

$$\mathring{\mu}_n(d\,|\,e)_j = \inf_{A \in \mathfrak{I}_n(d,e)_j} \mu_n(A\,|\,e)_j.$$

Problem 1.1. Determine $\mathring{\mu}_n(d\,|\,e)_j$ in terms of n, d, e, j.

Schmidt [1967, Theorems 12–13, 15–17] proved several bounds on the quantity $\mathring{\mu}_n(d\,|\,e)_j$. In all what follows, let $t = \min(d, e)$.

Theorem 1.2 [Schmidt 1967]. *For all* $j \in \{1, \dots, t\}$, *one has*

$$\frac{d(n-j)}{j(n-d)(n-e)} \leqslant \mathring{\mu}_n(d\,|\,e)_j \leqslant \frac{1}{j}\left\lceil \frac{e(n-e)+1}{n+1-d-e} \right\rceil,$$

Moreover, when $j = 1$,

$$\mathring{\mu}_n(d\,|\,e)_1 \geqslant \frac{n(n-1)}{(n-d)(n-e)}.$$

Schmidt improved the lower bound when an additional hypothesis is met. He also determined some exact values of $\mathring{\mu}_n(d\,|\,e)_j$. In particular, Problem 1.1 is completely solved when $\min(d, e) = 1$.

Theorem 1.3 [Schmidt 1967]. *Let $j \in \{1, \dots, t\}$. If*

$$j + n - t \geqslant j(j + n - d - e),$$

then

$$\mathring{\mu}_n(d\,|\,e)_j \geqslant \frac{j + n - t}{j(j + n - d - e)}.$$

Moreover, when $j = t$,

$$\mathring{\mu}_n(d\,|\,e)_t = \frac{n}{t(t + n - d - e)}.$$

A direct application of Schmidt's going-up theorem [1967, Theorem 9] is the following result proved in Section 5 below.

Proposition 1.4. *Let $d, e, j, \ell \in \mathbb{N}^*$ be such that $d + e \leqslant n$, $1 \leqslant j \leqslant \ell \leqslant e$ and $j \leqslant d$. Then*

$$\mathring{\mu}_n(d\,|\,e)_j \geqslant \frac{n - \ell}{n - e} \cdot \mathring{\mu}_n(d\,|\,\ell)_j.$$

This proposition implies some straightforward improvements. For instance, the known lower bound $\mathring{\mu}_6(3\,|\,3)_2 \geqslant \frac{5}{4}$ (Theorem 1.2) becomes $\mathring{\mu}_6(3\,|\,3)_2 \geqslant \frac{4}{3}$ using $\mathring{\mu}_6(3\,|\,2)_2 = 1$ (Theorem 1.3).

Both N. Moshchevitin [2020, Satz 2] and N. de Saxcé [2020, Theorem 9.3.2] improved some upper bounds.

Theorem 1.5 [Moshchevitin 2020]. *Let $d \geqslant 1$ be an integer. One has*

$$\mathring{\mu}_{2d}(d\,|\,d)_1 \leqslant 2d.$$

Theorem 1.6 [de Saxcé 2020]. *Let $n \geqslant 2$ and $d \in \{1, \dots, \lfloor n/2 \rfloor\}$. One has*

$$\mathring{\mu}_n(d\,|\,d)_d \leqslant \frac{n}{d(n - d)}.$$

The simplest unknown case and also the last unknown case in $\mathbb{R}^4$ is $(n, d, e, j) = (4, 2, 2, 1)$. Theorem 1.2 together with Theorem 1.5 gives $3 \leqslant \mathring{\mu}_4(2\,|\,2)_1 \leqslant 4$. Here, we will show the following theorem.

Theorem 1.7. *One has*

$$\mathring{\mu}_4(2\,|\,2)_1 = 3.$$

The next unknown cases are in $\mathbb{R}^5$. One can notice that Theorem 1.2 combined with Theorem 1.3 gives $4 \leqslant \mathring{\mu}_5(3\,|\,2)_1 \leqslant 7$. This upper bound is improved by 1.

Theorem 1.8. *One has*

$$\mathring{\mu}_5(3\,|\,2)_1 \leqslant 6.$$

Combining Theorem 1.5 and Proposition 1.4, an improvement on the known bound for $\mathring{\mu}_{2d}(d\,|\,\ell)_1$ is deduced; see the beginning of Section 5 for examples.

Theorem 1.9. *Let $d \geqslant 2$ and $\ell \in \{1, \dots, d\}$. One has*

$$\mathring{\mu}_{2d}(d\,|\,\ell)_1 \leqslant \frac{2d^2}{2d - \ell}.$$

Finally, we prove a new lower bound in the general case.

Theorem 1.10. *Let $n \geqslant 4$ and $d, e \in \{1, \dots, n-1\}$ such that $d + e \leqslant n$; let $j \in \{1, \dots, \min(d, e)\}$. One has*

$$\mathring{\mu}_n(d\,|\,e)_j \geqslant \frac{(n-j)(jn - jd + j^2/2 + j/2 + 1)}{j^2(n-e)(n-d+j/2+1/2)}.$$

This leads to the following corollary.

Corollary 1.11. *One has, for any fixed $d \geqslant 1$,*

$$\lim_{n\to+\infty} \mathring{\mu}_n(d\,|\,d)_d = \frac{1}{d}.$$

Section 2 focuses on the case of the approximation of a plane by rational planes in $\mathbb{R}^4$ (Theorem 1.7). In Section 3 we approximate a subspace of dimension 3 by rational planes (Theorem 1.8). Then, in Section 4, we comment briefly on the method developed in the previous two sections. Section 5 contains a proof of Theorem 1.9. Finally, Section 6 develops how to decompose the subspace one wants to approach into subspaces of lower dimensions, and this leads to a proof of Theorem 1.10 and Corollary 1.11.

2. Approximation of a plane by rational planes in $\mathbb{R}^4$

The main result is Theorem 1.7: $\mathring{\mu}_4(2|2)_1 = 3$. It finishes the solution of Problem 1.1 for $n \leqslant 4$. To prove this theorem, some planes of $\mathbb{R}^4$ are explicitly constructed, which are $(2, 1)$-irrational and not so well approximated by rational planes. For $\xi \in]0, \sqrt{7}[$, let us consider the plane A_ξ of $\mathbb{R}^4$ spanned by

$$X_\xi^{(1)} = \begin{pmatrix} 0 \\ 1 \\ \xi \\ \sqrt{7-\xi^2} \end{pmatrix} \quad \text{and} \quad X_\xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -\sqrt{7-\xi^2} \\ \xi \end{pmatrix}.$$

The crucial lemma in order to prove Theorem 1.7 is Lemma 2.1 below, which requires the function φ,

$$\varphi(A, B) = \prod_{j=1}^{\min(\dim A, \dim B)} \psi_j(A, B). \tag{1}$$

Lemma 2.1. *There exist real numbers $\xi \in]0, \sqrt{7}[$ and $c > 0$ such that $A_\xi \in \mathfrak{I}_4(2, 2)_1$ and, for all $B \in \mathfrak{R}_4(2)$,*

$$\varphi(A_\xi, B) \geqslant \frac{c}{H(B)^3}. \tag{2}$$

From Lemmas Lemma 2.1 and 2.6 below, we shall deduce the following proposition.

Proposition 2.2. *There exists $\xi \in]0, \sqrt{7}[$ such that*

$$\mu_4(A_\xi\,|\,2)_1 = 3.$$

Theorem 1.7 comes directly from the definition of $\mathring{\mu}$, Proposition 2.2 and Theorem 1.2. Before proving Proposition 2.2, let us introduce some notation and two basic lemmas.

Given vectors $X_1, \dots, X_e \in \mathbb{R}^n$, let us denote by $M \in \mathrm{M}_{n,e}(\mathbb{R})$ the matrix whose j-th column is X_j for $j \in \{1, \dots, e\}$. Let us define the *generalised determinant* of the family $(X_1, \dots, X_e)$ to be

$D(X_1, \dots, X_e) = \sqrt{\det({}^t MM)}$. The following result gives an equivalent definition of the height of a rational subspace (see [Schmidt 1991, Lemma 5H and Corollary 5I]).

Theorem 2.3. *Let $B \in \mathfrak{R}_n(e)$ and $(X_1, \dots, X_e)$ be a basis of $B \cap \mathbb{Z}^n$. If one denotes by $\eta = (\eta_1, \dots, \eta_N)$, where $N = \binom{n}{e}$, the Plücker coordinates associated with $(X_1, \dots, X_e)$ and ordered by lexicographic order, one has $\eta \in \mathbb{Z}^N$ and $\gcd(\eta_1, \dots, \eta_N) = 1$. Moreover,*

$$H(B) = D(X_1, \dots, X_e).$$

Let us make a link between proximity and height.

Lemma 2.4. *Let $n \geqslant 2$, $d, e \in \{1, \dots, n-1\}$ be such that $d + e = n$, A be a subspace of dimension d of $\mathbb{R}^n$ and $B \in \mathfrak{R}_n(e)$. Let $(X_1, \dots, X_d)$ be a basis of A, $(Y_1, \dots, Y_e)$ be a basis of $B \cap \mathbb{Z}^n$, and denote by $M \in \mathrm{M}_n(\mathbb{R})$ the matrix whose columns are $X_1, \dots, X_d, Y_1, \dots, Y_e$ respectively. There exists a constant $c > 0$ depending only on $(X_1, \dots, X_d)$ such that*

$$\varphi(A, B) = c \frac{|\det M|}{H(B)}.$$

Proof. The following claim comes from equation (7), page 446 of [Schmidt 1967].

Claim 2.5. *One has*

$$\varphi(A, B) = \frac{D(X_1, \dots, X_d, Y_1, \dots, Y_e)}{D(X_1, \dots, X_d) D(Y_1, \dots, Y_e)}.$$

Since $(Y_1, \dots, Y_e)$ is a basis of $B \cap \mathbb{Z}^n$, Claim 2.5 together with Theorem 2.3 gives us

$$\varphi(A, B) = c D(X_1, \dots, X_d, Y_1, \dots, Y_e) H(B)^{-1},$$

where $c = D(X_1, \dots, X_d)^{-1} > 0$ is a constant depending only on $(X_1, \dots, X_d)$. Moreover, the matrix M is a square matrix, so $D(X_1, \dots, X_d, Y_1, \dots, Y_e)^2 = \det({}^t MM) = \det(M)^2$. Thereby, since $D(X_1, \dots, X_d, Y_1, \dots, Y_e) \geqslant 0$, one has $\varphi(A, B) = c |\det M| H(B)^{-1}$. □

Lemma 2.6. *Let $n \geqslant 2$, A and B be two subspaces of $\mathbb{R}^n$ of dimensions d and e respectively. Then for all $j \in \{1, \dots, \min(d, e)\}$, we have $\psi_j(A, B) \geqslant \varphi(A, B)^{1/j}$.*

Proof. Let $t = \min(d, e)$ and $j \in \{1, \dots, t\}$. From the definition of the ψ_i, one has $\psi_1(A, B) \leqslant \cdots \leqslant \psi_t(A, B) \leqslant 1$. Thereby, the product in (1) can be split as

$$\varphi(A, B) = \left(\prod_{i=1}^{j} \underbrace{\psi_i(A, B)}_{\leqslant \psi_j(A, B)} \right) \times \left(\prod_{i=j+1}^{t} \underbrace{\psi_i(A, B)}_{\leqslant 1} \right) \leqslant \psi_j(A, B)^j. \qquad \square$$

We can now provide a proof of Proposition 2.2.

Proof of Proposition 2.2. Together with Lemma 2.6 applied for $j = 1$, Lemma 2.1 shows that $\mu_4(A_\xi | 2)_1 \leqslant 3$. Since Theorem 1.2 gives $\mu_4(A_\xi | 2)_1 \geqslant \mathring{\mu}_4(2 | 2)_1 \geqslant 3$, Proposition 2.2 follows. □

In order to prove Lemma 2.1, we will use the following definition and theorem (see [Beresnevich 2015, Corollary 1]).

Definition 2.7. Let **Bad** be the set of all $y \in \mathbb{R}^k$ such that there exists $c > 0$ such that the only integer solution $(a_0, \dots, a_k)$ to the inequality

$$|a_0 + a_1 y_1 + \cdots + a_k y_k| < c \|(a_1, \dots, a_k)\|_\infty^{-k}$$

is the trivial one $(0, \dots, 0)$.

Theorem 2.8 (Beresnevich, 2015). *Let $\mathcal{M}$ be a manifold immersed into $\mathbb{R}^n$ by an analytic nondegenerate map. Then* **Bad** $\cap \mathcal{M}$ *has the same Hausdorff dimension as $\mathcal{M}$; in particular* **Bad** $\cap \mathcal{M} \neq \varnothing$.

Finally, let us prove Lemma 2.1.

Proof of Lemma 2.1. Let $B \in \mathfrak{R}_4(2)$ and (Y_1, Y_2) be a basis of $B \cap \mathbb{Z}^4$. Let us denote by $(\eta_1, \dots, \eta_6)$ a set of Plücker coordinates of B associated with the basis (Y_1, Y_2) as in Theorem 2.3, so that $(\eta_1, \dots, \eta_6) \in \mathbb{Z}^6$ and $\gcd(\eta_1, \dots, \eta_6) = 1$. Moreover, this vector satisfies the Plücker relation (see [Caldero and Germoni 2015, Theorem 2.9]) for a subspace of dimension 2 of $\mathbb{R}^4$:

$$\eta_1\eta_6 - \eta_2\eta_5 + \eta_3\eta_4 = 0. \tag{3}$$

The manifold $\mathcal{M} = \{(1, \xi, \sqrt{7-\xi^2}) : \xi \in]0, \sqrt{7}[\}$ is nondegenerate (the functions $\xi \mapsto 1$, $\xi \mapsto \xi$, and $\xi \mapsto \sqrt{7-\xi^2}$ are linearly independent over $\mathbb{R}$), so Theorem 2.8 implies the existence of $\xi \in]0, \sqrt{7}[$ such that $(1, \xi, \sqrt{7-\xi^2}) \in$ **Bad**. In particular 1, ξ and $\sqrt{7-\xi^2}$ are linearly independent over $\mathbb{Q}$. Let us denote by M_ξ the matrix of $\mathrm{M}_4(\mathbb{R})$ whose columns are $X_\xi^{(1)}, X_\xi^{(2)}, Y_1, Y_2$ respectively. Notice that $A_\xi \cap B = \{0\}$ if, and only if, $\det M_\xi \neq 0$. The determinant of M_ξ is computed by a Laplace expansion on its two first columns:

$$\det M_\xi = -\eta_6 + \eta_5\xi - \eta_4\sqrt{7-\xi^2} - \eta_3\sqrt{7-\xi^2} - \eta_2\xi + 7\eta_1. \tag{4}$$

Assuming that $\det M_\xi = 0$ we have

$$-\eta_6 + 7\eta_1 + (\eta_5 - \eta_2)\xi + (-\eta_3 - \eta_4)\sqrt{7-\xi^2} = 0. \tag{5}$$

Since $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(1, \xi, \sqrt{7-\xi^2}) = 3$ and the η_i are integers, (5) gives

$$(\eta_4, \eta_5, \eta_6) = (-\eta_3, \eta_2, 7\eta_1). \tag{6}$$

Thereby, (3) becomes

$$\eta_2^2 + \eta_3^2 = 7\eta_1^2.$$

Reducing modulo 4, this equation implies that η_1, η_2 and η_3 are even, which contradicts the assumption $\gcd(\eta_1, \dots, \eta_6) = 1$ using (6). Thereby, $\det M_\xi \neq 0$, so $A_\xi \cap B = \{0\}$ which proves that the subspace A_ξ is $(2, 1)$-irrational.

To establish inequality (2) of Lemma 2.1, recall that the basis (Y_1, Y_2) of B is also a $\mathbb{Z}$-basis of $B \cap \mathbb{Z}^4$. Hence, Lemma 2.4 gives a constant $c_1 > 0$ depending only on $(X_\xi^{(1)}, X_\xi^{(2)})$ such that

$$\varphi(A_\xi, B) = |\det(M_\xi)| \frac{c_1}{H(B)}. \tag{7}$$

Since the Plücker coordinates $\eta = (\eta_1, \dots, \eta_6)$ of B are integers and satisfy $\gcd(\eta_1, \dots, \eta_6) = 1$, one has

$$H(B) = \|\eta\|. \tag{8}$$

Now recall that we have chosen ξ in such a way that there exists a constant $c_2 > 0$ such that for all $q = (a, b, c) \in \mathbb{Z}^3 \backslash \{(0, 0, 0)\}$

$$|a\sqrt{7-\xi^2} + b\xi + c| \geqslant c_2 \|q\|^{-2}. \tag{9}$$

Notice that for $q = (-\eta_3 - \eta_4, \eta_5 - \eta_2, -\eta_6 + 7\eta_1)$, one has $q \neq (0, 0, 0)$, otherwise (6) would be true, and it was already said that this was impossible. Moreover, $\|q\| \leqslant \sqrt{67}\|\eta\|$, so inequality (9) combined with (4) gives

$$|\det(M_\xi)| \geqslant c_3 \|\eta\|^{-2}.$$

This inequality together with (7) and (8) give a constant $c_4 > 0$ such that

$$\varphi(A_\xi, B) \geqslant \frac{c_4}{H(B)^3}. \qquad \square$$

Remark 2.9. In the same way, one can construct infinitely many subspaces A_ξ defined over $\overline{\mathbb{Q}}$ satisfying $\mu_4(A_\xi|2)_1 = 3$ with a theorem of Schmidt. The point is to replace in the proof of Lemma 2.1 the use of Theorem 2.8 by Theorem 2 of [Schmidt 1970]; the only difference is that the exponent -2 in (9) becomes $-2-\varepsilon$ for any $\varepsilon > 0$, and -3 becomes $-3-\varepsilon$ in (2). Up to this modification, Lemma 2.1 and Proposition 2.2 are still true if $\xi \in]0, \sqrt{7}[$ is a real algebraic number satisfying $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(1, \xi, \sqrt{7-\xi^2}) = 3$. In particular, for $\xi = \sqrt{2}$, one gets the explicit example

$$\mu_4(A_{\sqrt{2}}|2)_1 = 3.$$

3. Approximation of a subspace of dimension 3 by rational planes in $\mathbb{R}^5$

The method developed here is very similar to the one used in Section 2, so we will not linger on the details in this section. Computations are not detailed, see [Joseph 2021] for extended computations. The main result is Theorem 1.8: $\mathring{\mu}_5(3|2)_1 \leqslant 6$.

As in Section 2, a subspace of $\mathbb{R}^5$ is explicitly constructed so that it is (2, 1)-irrational and at the same time not so well approximated by rational planes of $\mathbb{R}^5$. We will start by stating some lemmas to prove this statement; the proofs of the lemmas will follow later.

Let ζ_3 be a real number, let us consider the four real numbers

$$\zeta_1 = -\frac{112\zeta_3^4 - 196\zeta_3^3 - (42\sqrt{2}\zeta_3^3 - 17\sqrt{2}\zeta_3^2 + 13\sqrt{2}\zeta_3)\sqrt{4\zeta_3 - 5}\sqrt{\zeta_3 - 1} + 88\zeta_3^2 - 30\zeta_3 + 6}{4\big(10\zeta_3^4 - 7\zeta_3^3 - (4\sqrt{2}\zeta_3^3 + 3\sqrt{2}\zeta_3^2 + \sqrt{2})\sqrt{4\zeta_3 - 5}\sqrt{\zeta_3 - 1} - 10\zeta_3^2 + 5\zeta_3 - 2\big)},$$

$$\zeta_2 = -\frac{52\zeta_3^4 - 154\zeta_3^3 - (18\sqrt{2}\zeta_3^3 - 35\sqrt{2}\zeta_3^2 + 13\sqrt{2}\zeta_3 - 6\sqrt{2})\sqrt{4\zeta_3 - 5}\sqrt{\zeta_3 - 1} + 148\zeta_3^2 - 60\zeta_3 + 18}{4\big(10\zeta_3^4 - 7\zeta_3^3 - (4\sqrt{2}\zeta_3^3 + 3\sqrt{2}\zeta_3^2 + \sqrt{2})\sqrt{4\zeta_3 - 5}\sqrt{\zeta_3 - 1} - 10\zeta_3^2 + 5\zeta_3 - 2\big)},$$

$$\zeta_4 = -\frac{\sqrt{2}\sqrt{4\zeta_3 - 5}\sqrt{\zeta_3 - 1}\zeta_3^2 - 6\zeta_3^3 + 3\zeta_3^2 + 3\zeta_3}{2(\zeta_3^2 - 1)},$$

$$\zeta_5 = -\frac{\sqrt{2}\sqrt{4\zeta_3 - 5}\sqrt{\zeta_3 - 1}\zeta_3 - 3\zeta_3^2 + 3\zeta_3}{2(\zeta_3^2 - 1)},$$

assuming $\zeta_3 \geqslant \frac{5}{4}$ so that all square roots are well defined, and ζ_3 large enough so that all denominators are nonzero (actually, $\zeta_3 \geqslant \frac{5}{4}$ is sufficient for both conditions). Let

$$\begin{aligned} &\xi_1 = 1, && \xi_2 = \zeta_2 + \zeta_5, && \xi_3 = -\zeta_1, && \xi_4 = 1 + \zeta_1 + \zeta_5, && \xi_5 = \zeta_2, \\ &\xi_6 = 2\zeta_2 - \zeta_5, && \xi_7 = -\zeta_3, && \xi_8 = \zeta_3, && \xi_9 = \zeta_4, && \xi_{10} = \zeta_5 \end{aligned}$$

and finally $\xi = (\xi_1, \dots, \xi_{10})$. The following lemma allows us to construct the subspace of $\mathbb{R}^5$ wanted.

Lemma 3.1. *There exists a subspace A_ξ of dimension 3 of $\mathbb{R}^5$ which admits the vector ξ as Plücker coordinates (with respect to lexicographic order).*

Now that the subspace A_ξ has been constructed, we can state that it is indeed (2, 1)-irrational and not so well approximated by rational planes of $\mathbb{R}^5$.

Lemma 3.2. *There exist real numbers $\zeta_3 \geqslant \frac{5}{4}$ and $c > 0$ such that $A_\xi \in \mathfrak{I}_5(3,2)_1$ and, for all $B \in \mathfrak{R}_5(2)$,*

$$\varphi(A_\xi, B) \geqslant \frac{c}{H(B)^6}. \tag{10}$$

This lemma together with Lemma 2.6 immediately leads to the following proposition.

Proposition 3.3. *There exists $\zeta_3 \geqslant \frac{5}{4}$ such that*

$$\mu_5(A_\xi|2)_1 \leqslant 6.$$

Much as in Section 2, Theorem 1.8 is an immediate consequence of Proposition 3.3, which itself follows from Lemma 2.6 and Lemma 3.2. We will start with the proof of Lemma 3.1.

Proof of Lemma 3.1. There exists a subspace which admits ξ as Plücker coordinates if, and only if, the coordinates of ξ satisfy the Plücker relations (see [Caldero and Germoni 2015, Theorem 2.9]) for a subspace of dimension 3 of $\mathbb{R}^5$:

$$\begin{cases} \xi_2\xi_5 = \xi_3\xi_4 + \xi_1\xi_6, \\ \xi_2\xi_8 = \xi_3\xi_7 + \xi_1\xi_9, \\ \xi_4\xi_8 = \xi_5\xi_7 + \xi_1\xi_{10}, \\ \xi_4\xi_9 = \xi_6\xi_7 + \xi_2\xi_{10}, \\ \xi_5\xi_9 = \xi_6\xi_8 + \xi_3\xi_{10}. \end{cases} \tag{11}$$

A basic formal computation shows that the vector ξ, as it has been defined, indeed satisfies system (11). □

Before proving the crucial Lemma 3.2, we need a technical result.

Lemma 3.4. *The manifold $\mathcal{M} = \{(1, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5) : \zeta_3 \geqslant \frac{5}{4}\}$ is nondegenerate.*

Proof. Let $(a_0, \dots, a_5) \in \mathbb{R}^6$ such that $a_0 + a_1\zeta_1 + \cdots + a_5\zeta_5 = 0$ for any $\zeta_3 \geqslant \frac{5}{4}$. One can compute polynomials $P_1, P_2, P_3 \in \mathbb{R}[X]$ such that

$$0 = a_0 + a_1\zeta_1 + \cdots + a_5\zeta_5 = \frac{P_1(\zeta_3) + P_2(\zeta_3)\sqrt{P_3(\zeta_3)}}{10\zeta_3^3 + 7\zeta_3 - 2 - (4\zeta_3^2 - \zeta_3 + 1)\sqrt{P_3(\zeta_3)}}.$$

Hence, one has $P_1(\zeta_3) + P_2(\zeta_3)\sqrt{P_3(\zeta_3)} = 0$, so for all $\zeta_3 \geqslant \frac{5}{4}$ we have $P(\zeta_3) = P_1^2(\zeta_3) - P_2^2(\zeta_3)P_3(\zeta_3) = 0$. The four equations given by the monomials of degrees 32, 30, 28 and 26 lead to a system of equations between the a_i, which implies $a_0 = a_3 = a_4 = a_5$. Considering the monomials of degree 22 lead to $14a_1^2 + 4a_1a_2 - a_2^2 = 0$, so $a_2 = (2 \pm 3\sqrt{2})a_1$, and the monomials of degree 21 lead to $7a_1^2 - 118a_1a_2 + 19a_2^2 = 0$ which cannot be. Therefore, $a_i = 0$ for all $i \in \{0, \dots, 5\}$ so the manifold considered is nondegenerate. □

With Lemma 3.4, we are now able to prove Lemma 3.2. Notice that the proof is quite similar to the proof of Lemma 2.1.

Proof of Lemma 3.2. Let $B \in \mathfrak{R}_5(2)$ and (Y_1, Y_2) be a basis of $B \cap \mathbb{Z}^5$, let us denote by $(\eta_1, \dots, \eta_{10})$ a set of Plücker coordinates for B associated with the basis (Y_1, Y_2) ordered by lexicographic order. According to Theorem 2.3, one has $(\eta_1, \dots, \eta_{10}) \in \mathbb{Z}^{10}$ and $\gcd(\eta_1, \dots, \eta_{10}) = 1$. Moreover, this vector satisfies the Plücker relations for a subspace of dimension 2 of $\mathbb{R}^5$:

$$\begin{cases} \eta_2\eta_5 = \eta_3\eta_4 + \eta_1\eta_6, \\ \eta_2\eta_8 = \eta_3\eta_7 + \eta_1\eta_9, \\ \eta_4\eta_8 = \eta_5\eta_7 + \eta_1\eta_{10}, \\ \eta_4\eta_9 = \eta_6\eta_7 + \eta_2\eta_{10}, \\ \eta_5\eta_9 = \eta_6\eta_8 + \eta_3\eta_{10}. \end{cases} \tag{12}$$

According to Lemma 3.4, the manifold $\mathcal{M} = \left\{(1, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5) : \zeta_3 \geqslant \frac{5}{4}\right\}$ is nondegenerate, so Theorem 2.8 implies the existence of $\zeta_3 \geqslant \frac{5}{4}$ such that $(1, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5) \in$ **Bad**. In particular, $1, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5$ are linearly independent over $\mathbb{Q}$. Let $(X_\xi^{(1)}, X_\xi^{(2)}, X_\xi^{(3)})$ be a basis of A_ξ associated with ξ. Let us denote by M_ξ the matrix of $\mathrm{M}_5(\mathbb{R})$ whose columns are $X_\xi^{(1)}, X_\xi^{(2)}, X_\xi^{(3)}, Y_1, Y_2$ respectively. Notice that $A_\xi \cap B = \{0\}$ if, and only if, $\det M_\xi \neq 0$. The determinant of M_ξ is computed by a Laplace expansion on its first three columns:

$$\det M_\xi = \xi_1\eta_{10} - \xi_2\eta_9 + \xi_3\eta_8 + \xi_4\eta_7 - \xi_5\eta_6 + \xi_6\eta_5 - \xi_7\eta_4 + \xi_8\eta_3 - \xi_9\eta_2 + \xi_{10}\eta_1.$$

Let us assume that $\det M_\xi = 0$; this implies

$$\begin{aligned} 0 &= \det(M_\xi) \\ &= \eta_{10} - (\zeta_2 + \zeta_5)\eta_9 - \zeta_1\eta_8 + (1 + \zeta_1 + \zeta_5)\eta_7 - \zeta_2\eta_6 + (2\zeta_2 - \zeta_5)\eta_5 + \zeta_3\eta_4 + \zeta_3\eta_3 - \zeta_4\eta_2 + \zeta_5\eta_1 \\ &= \eta_{10} + \eta_7 + (-\eta_8 + \eta_7)\zeta_1 + (-\eta_9 - \eta_6 + 2\eta_5)\zeta_2 + (\eta_4 + \eta_3)\zeta_3 - \eta_2\zeta_4 + (-\eta_9 + \eta_7 - \eta_5 + \eta_1)\zeta_5. \end{aligned}$$

Since $1, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5$ are linearly independent over $\mathbb{Q}$ and the η_i are integers, the equation above yields the relations

$$(\eta_1, \eta_2, \eta_4, \eta_6, \eta_8, \eta_{10}) = (\eta_9 - \eta_7 + \eta_5, 0, -\eta_3, -\eta_9 + 2\eta_5, \eta_7, -\eta_7).$$

Thus, system (12) becomes

$$\begin{cases} \eta_3^2 - 2\eta_5^2 + 2\eta_5\eta_7 - \eta_5\eta_9 - \eta_7\eta_9 + \eta_9^2 = 0, \\ -\eta_3\eta_7 - \eta_5\eta_9 + \eta_7\eta_9 - \eta_9^2 = 0, \\ -\eta_3\eta_7 - \eta_7^2 + \eta_7\eta_9 = 0, \\ -2\eta_5\eta_7 - \eta_3\eta_9 + \eta_7\eta_9 = 0, \\ \eta_3\eta_7 - 2\eta_5\eta_7 + \eta_5\eta_9 + \eta_7\eta_9 = 0, \end{cases} \tag{13}$$

whose set of rational solutions is the singleton $\{(0, \dots, 0)\}$ (once again, the computations can be found in [Joseph 2021]). Thereby, $\det M_\xi \neq 0$, so $A_\xi \cap B = \{0\}$ which implies that $A_\xi \in \mathfrak{I}_5(3, 2)_1$.

The proof of second part of the lemma is almost identical to the proof of (2) in Lemma 2.1, but with 6 reals numbers instead of 3. □

Remark 3.5. Similarly as in Section 2, one can construct infinitely many subspaces A_ξ defined over $\overline{\mathbb{Q}}$ satisfying $\mu_5(A_\xi \,|\, 2)_1 \leqslant 6$ with Theorem 2 of [Schmidt 1970]. The only difference is that the exponent -6 in (10) becomes $-6 - \varepsilon$ for any $\varepsilon > 0$. Up to this modification, Lemma 3.2 and Proposition 3.3 are still true if $\zeta_3 \geqslant \frac{5}{4}$ is a real algebraic number satisfying $[\mathbb{Q}(\zeta_3) : \mathbb{Q}] \geqslant 33$.

4. Some comments on the method

We believe that the method developed in Sections 2 and 3 can be used to improve several other upper bounds for $\mathring{\mu}_n(d\,|\,e)_1$ when $d+e=n$. As one can see in Section 3, the computations seem to be significantly more complicated with n growing. The main difficulty in $\mathbb{R}^5$ was to construct a subspace A_ξ complicated enough so that system (13) would not have any nontrivial rational solution — which implies $A_\xi \in \mathfrak{I}_5(3,2)_1$ — but also sufficiently simple so that it is indeed possible to show that this system does not have any nontrivial rational solution.

This method creates two contradictory wishes on the subspace A desired:

- to have *a lot* of Plücker coordinates linearly independent over $\mathbb{Q}$ so that A is $(e,1)$-irrational;
- to have *few* Plücker coordinates linearly independent over $\mathbb{Q}$ to obtain the best possible exponent with Theorem 2.8.

5. Application of Schmidt's going-up theorem

Here, we will prove Corollary 5.2 which implies Proposition 1.4 from which is immediately deduced Theorem 1.9: $\mathring{\mu}_{2d}(d\,|\,\ell)_1 \leqslant 2d^2/(2d-\ell)$. Indeed, Proposition 1.4 together with Theorem 1.5 gives, for $\ell \in \{1,\dots,d\}$, $\mathring{\mu}_{2d}(d\,|\,\ell)_1 \leqslant (2d-d)/(2d-\ell)\mathring{\mu}_{2d}(d\,|\,d)_1 \leqslant 2d^2/(2d-\ell)$.

Theorem 1.9 allows us to improve on numerous known upper bounds for $\mathring{\mu}_{2d}(d\,|\,\ell)_1$, since for instance taking $\ell = d-1$ implies

$$\frac{2d^2}{2d-\ell} \underset{d\to+\infty}{\sim} 2d$$

and the known upper bound for $\mathring{\mu}_{2d}(d\,|\,d-1)_1$, given by Theorem 1.2, is asymptotically equivalent to $\lfloor d^2/2 \rfloor$. Notice that when ℓ is fixed and d tends to $+\infty$, Theorem 1.2 gives an upper bound asymptotically equivalent to 2ℓ, which is better than our new bound. The best improvements occur when ℓ is close to d, for instance Theorem 1.9 implies $\mathring{\mu}_6(3\,|\,2)_1 \leqslant \frac{9}{2}$ improving on $\mathring{\mu}_6(3\,|\,2)_1 \leqslant 5$, $\mathring{\mu}_{12}(6\,|\,4)_1 \leqslant 9$ improving on $\mathring{\mu}_{12}(6\,|\,4)_1 \leqslant 11$, and $\mathring{\mu}_{22}(11\,|\,6)_1 \leqslant 15.125$ improving on $\mathring{\mu}_{12}(6\,|\,4)_1 \leqslant 17$.

Let us now state Schmidt's going-up theorem [1967, Theorem 9].

Theorem 5.1 (going-up [Schmidt 1967]). *Let $d, e \in \mathbb{N}^*$ be such that $d+e<n$; let $t=\min(d,e)$. Let A be a subspace of $\mathbb{R}^n$ of dimension d and $B \in \mathfrak{R}_n(e)$. Let $H \geqslant 1$ be such that $H(B) \leqslant H$, and such that there exist $x_i, y_i \in \mathbb{R}$ such that for all $i \in \{1,\dots,t\}$, $H(B)^{x_i}\psi_i(A,B) \leqslant c_1 H^{-y_i}$ with $c_1>0$. Then there exists a constant $c_2>0$ depending only on n and e, and a constant $c_3>0$ depending only of n, e, x_i and y_i, such that if $H' = c_2 H^{(n-e-1)/(n-e)}$, then there exists $C \in \mathfrak{R}_n(e+1)$ such that $C \supset B$, $H(C) \leqslant H'$ and, for all $i \in \{1,\dots,t\}$,*

$$H(C)^{x_i(n-e)/(n-e-1)}\psi_i(A,C) \leqslant c_1 c_3 H'^{-y_i(n-e)/(n-e-1)}.$$

Let us formulate a corollary to the going-up theorem.

Corollary 5.2. *Let $d, e, j, \ell \in \mathbb{N}^*$ be such that $d+e \leqslant n$, $1 \leqslant j \leqslant \ell \leqslant e$ and $j \leqslant d$. Then for all $A \in \mathfrak{I}_n(d,e)_j$, one has $A \in \mathfrak{I}_n(d,\ell)_j$ and*

$$\mu_n(A\,|\,e)_j \geqslant \frac{n-\ell}{n-e} \cdot \mu_n(A\,|\,\ell)_j.$$

Since $\mathfrak{I}_n(d,e)_j \subset \mathfrak{I}_n(d,\ell)_j$, Corollary 5.2 implies immediately Proposition 1.4 stated in the Introduction.

Remark 5.3. Notice that Corollary 5.2 generalises Theorem 2 of [Laurent 2009]. Corollary 5.2 does not necessarily need to be applied to a line, and the irrationality hypothesis is weaker than the one in [Laurent 2009].

Proof of Corollary 5.2. Notice that $\mathfrak{I}_n(d, e)_j \subset \mathfrak{I}_n(d, \ell)_j$ since $\ell \leqslant e$. Let $\alpha = \mu_n(A \,|\, \ell)_j$ and $\varepsilon > 0$; there exist infinitely many subspaces $B \in \mathfrak{R}_n(\ell)$ such that

$$\psi_j(A, B) \leqslant \frac{1}{H(B)^{\alpha-\varepsilon}}. \tag{14}$$

For each such subspace B, the going-up theorem applied $e - \ell$ times gives a subspace $C \in \mathfrak{R}_n(e)$ such that $C \supset B$ and

$$\psi_j(A, C) \leqslant \frac{c}{H(C)^{(\alpha-\varepsilon)(n-\ell)/(n-e)}}, \tag{15}$$

with $c > 0$ depending only on A and ε. The subspace A is (e, j)-irrational, so for all $C \in \mathfrak{R}_n(e)$, $\psi_j(A, C) \neq 0$. Thus, if there were only a finite number of rational subspaces C such that inequality (15) holds, there would be a constant $c' > 0$ such that, for all $C \in \mathfrak{R}_n(e)$,

$$\psi_j(A, C) > c'. \tag{16}$$

Since there are infinitely many subspaces $B \in \mathfrak{R}_n(\ell)$ such that inequality (14) holds, there exist such subspaces of arbitrary large height, thus such that $\psi_j(A, B) \leqslant c'$. The subspace C obtained from B with the going-up theorem satisfies $B \subset C$, so $\psi_j(A, C) \leqslant \psi_j(A, B) \leqslant c'$, which contradicts (16). Hence, there are infinitely many subspaces $C \in \mathfrak{R}_n(e)$ such that (15) holds, and the corollary follows. □

6. A lower bound for $\mathring{\mu}_n(d \,|\, e)_j$ in the general case

The goal here is to prove a new lower bound for $\mathring{\mu}_n(d \,|\, e)_j$ (Theorem 1.10). The strategy is to break down the subspace we want to approach into subspaces of lower dimension (here, we will use lines). It is then possible to approach simultaneously each line (it will be done with Dirichlet's approximation theorem), and to deduce an approximation of the original subspace.

The bound given by Theorem 1.10 improves asymptotically (for fixed j, d and e) the known lower bound for $\mathring{\mu}_n(d \,|\, e)_j$ (Theorem 1.2).

Let $d \leqslant n/2$. Combining Theorem 1.10 with Theorem 1.6, one obtains

$$\frac{2dn - d^2 + d + 2}{2d^2 n - d^3 + d^2} \leqslant \mathring{\mu}_n(d \,|\, d)_d \leqslant \frac{n}{d(n-d)},$$

and hence Corollary 1.11,

$$\lim_{n \to +\infty} \mathring{\mu}_n(d \,|\, d)_d = \frac{1}{d}.$$

The proof of Theorem 1.10 will require a lemma on the behaviour of the proximity function ψ with direct sums.

Lemma 6.1. *Let $n \geqslant 4$ and $F_1, \dots, F_\ell, B_1, \dots, B_\ell$ be 2ℓ subspaces of $\mathbb{R}^n$ such that, for all $i \in \{1, \dots, \ell\}$, $\dim F_i = \dim B_i = d_i$. Assume that the F_i span a subspace of dimension $k = d_1 + \cdots + d_\ell$ and so do* *the B_i. Let $F = F_1 \oplus \cdots \oplus F_\ell$ and $B = B_1 \oplus \cdots \oplus B_\ell$, then one has*

$$\psi_k(F, B) \leqslant c_{F,n} \sum_{i=1}^{\ell} \psi_{d_i}(F_i, B_i),$$

where $c_{F,n} > 0$ is a constant depending only on $F_1, \ldots, F_\ell$ and n.

Proof. The idea is to break down each F_i and each B_i into a direct sum of well chosen lines. For this, we will use the following claim.

Claim 6.2. *Let D and E be two subspaces of $\mathbb{R}^n$ of dimension k. There exist k lines $D_1, \ldots, D_k$ of D and k lines $E_1, \ldots, E_k$ of E such that $D = D_1 \oplus \cdots \oplus D_k$, $E = E_1 \oplus \cdots \oplus E_k$, and*

$$\psi_k(D, E) \leqslant \sum_{i=1}^{k} \psi_1(D_i, E_i) \leqslant k\psi_k(D, E). \tag{17}$$

Proof of Claim 6.2. There exist an orthonormal basis $(X_1, \ldots, X_k)$ of D and an orthonormal basis $(Y_1, \ldots, Y_k)$ of E such that for all $i \in \{1, \ldots, k\}$, $\psi_i(D, E) = \psi(X_i, Y_i)$. Moreover, for all $i \in \{1, \ldots, k\}$, one has $\psi_i(D, E) \leqslant \psi_k(D, E)$. Let us set, for $i \in \{1, \ldots, k\}$, $D_i = \operatorname{Span}(X_i)$ and $E_i = \operatorname{Span}(Y_i)$ to get the second part of inequality (17):

$$\sum_{i=1}^{k} \psi_1(D_i, E_i) = \sum_{i=1}^{k} \psi(X_i, Y_i) = \sum_{i=1}^{k} \psi_i(D, E) \leqslant k\psi_k(D, E).$$

The first part of inequality (17) is trivial since $\psi_1(D_i, E_i) \geqslant 0$ for any i, and $\psi_k(D, E) = \psi_1(D_k, E_k)$. □

We can come back to the proof of Lemma 6.1. Let $i \in \{1, \ldots, \ell\}$; according to Claim 6.2, there exist d_i lines $D_{i,1}, \ldots, D_{i,d_i}$ of F_i and d_i lines $E_{i,1}, \ldots, E_{i,d_i}$ of B_i such that

$$\sum_{j=1}^{d_i} \psi_1(E_{i,j}, D_{i,j}) \leqslant d_i \psi_{d_i}(F_i, B_i) \leqslant n\psi_{d_i}(F_i, B_i). \tag{18}$$

Let $a_{i,1}, \ldots, a_{i,d_i}$ be unitary vectors of $D_{i,1}, \ldots, D_{i,d_i}$ respectively and $b_{i,1}, \ldots, b_{i,d_i}$ be unitary vectors of $E_{i,1}, \ldots, E_{i,d_i}$ respectively, such that for all $j \in \{1, \ldots, d_i\}$, $a_{i,j} \cdot b_{i,j} \geqslant 0$. Let $(X_1, \ldots, X_k)$ and $(Y_1, \ldots, Y_k)$ be orthonormal bases of F and B respectively, such that $\psi_j(F, B) = \psi(X_j, Y_j)$ for any $j \in \{1, \ldots, k\}$. Let $Z = \lambda_1 Y_1 + \cdots + \lambda_k Y_k$ be a unitary vector of B. One has

$$|X_k \cdot Z| = \left|\sum_{i=1}^{k} \lambda_i X_k \cdot Y_i\right| \leqslant \sum_{i=1}^{k} |\lambda_i \delta_{i,k} X_k \cdot Y_i| \leqslant X_k \cdot Y_k$$

which implies

$$\psi_k(F, B) = \psi(X_k, Y_k) \leqslant \min_{Z \in B \setminus \{0\}} \psi(X_k, Z) = \psi_1(\operatorname{Span}(X_k), B).$$

Moreover, $\operatorname{Span}(Y_k) \subset B$, so $\psi_1(\operatorname{Span}(X_k), B) \leqslant \psi(X_k, Y_k)$. Hence

$$\psi_k(F, B) = \psi_1(\operatorname{Span}(X_k), B). \tag{19}$$

Let us decompose X_k in the basis $(a_{1,1}, \ldots, a_{\ell,d_\ell})$ as $X_k = \sum_{i=1}^{\ell} \sum_{j=1}^{d_i} x_{i,j} a_{i,j}$, and let

$$Y = \sum_{i=1}^{\ell} \sum_{j=1}^{d_i} x_{i,j} b_{i,j} \in B.$$

Since X_k is unitary, one has

$$\psi(X_k, Y) \leqslant \|X_k - Y\| = \left\| \sum_{i=1}^{\ell} \sum_{j=1}^{d_i} x_{i,j}(a_{i,j} - b_{i,j}) \right\| \leqslant \sum_{i=1}^{\ell} \sum_{j=1}^{d_i} |x_{i,j}| \|a_{i,j} - b_{i,j}\|,$$

where $\|\cdot\|$ stands for the Euclidean norm. For $i \in \{1, \ldots, \ell\}$ and $j \in \{1, \ldots, d_i\}$, let us consider the functions

$$p_{i,j} : F \to \mathbb{R}, \quad \sum_{i=1}^{\ell} \sum_{j=1}^{d_i} x_{i,j} a_{i,j} \mapsto x_{i,j}.$$

These functions are continuous on the compact $K = \{x \in F : \|x\| = 1\}$, so they are bounded on it. Thus, there exists $c^{(1)}_{F,n}$ a constant depending only on $a_{1,1}, \ldots, a_{\ell,d_\ell}$ such that for all $x = \sum_{i=1}^{\ell} \sum_{j=1}^{d_i} x_{i,j} a_{i,j} \in K$, one has $|x_{i,j}| \leqslant c^{(1)}_{F,n}$.

We now require an elementary claim.

Claim 6.3. *Let X and Y be unitary vectors such that $X \cdot Y \geqslant 0$. One has*

$$\psi(X, Y) \geqslant \frac{\sqrt{2}}{2} \|X - Y\|.$$

Proof. Let $p^{\perp}_{\mathrm{Span}(Y)}$ be the orthogonal projection onto $\mathrm{Span}(Y)$,

$$\alpha = \|X - p^{\perp}_{\mathrm{Span}(Y)}(X)\| \quad \text{and} \quad \beta = \|Y - p^{\perp}_{\mathrm{Span}(Y)}(X)\|.$$

One has $\|X - Y\|^2 = \alpha^2 + \beta^2$, and since X is unitary,

$$\psi(X, Y) = \psi(X, p^{\perp}_{\mathrm{Span}(Y)}(X)) = \|X - p^{\perp}_{\mathrm{Span}(Y)}(X)\| = \alpha.$$

Moreover, $X \cdot Y \geqslant 0$, so $1 = \|X\|^2 = (1 - \beta)^2 + \alpha^2$; hence there exists $\theta \in \left[0, \frac{\pi}{2}\right]$ such that $1 - \beta = \cos\theta$ and $\alpha = \sin\theta$. Since $1 - \cos\theta \leqslant \sin\theta$, we have $\beta \leqslant \alpha$, and finally $\|X - Y\|^2 \leqslant 2\alpha^2 = 2\psi(X, Y)^2$. □

We can come back to the proof of Lemma 6.1. Since for all i, j one has $a_{i,j} \cdot b_{i,j} \geqslant 0$, applying Claim 6.3 yields

$$\psi(X_k, Y) \leqslant c^{(1)}_{F,n} \sum_{i=1}^{\ell} \sum_{j=1}^{d_i} \|a_{i,j} - b_{i,j}\| \leqslant c^{(2)}_{F,n} \sum_{i=1}^{\ell} \sum_{j=1}^{d_i} \psi_1(D_{i,j}, E_{i,j})$$

because the $a_{i,j}$ and the $b_{i,j}$ are unitary vectors, with $c^{(2)}_{F,n} = \sqrt{2} c^{(1)}_{F,n}$. Finally, inequality (18) implies

$$\psi(X_k, Y) \leqslant c^{(2)}_{F,n} n \sum_{i=1}^{\ell} \psi_{d_i}(F_i, B_i) \tag{20}$$

and with (19) yields

$$\psi_k(F, B) \leqslant \psi_1(\mathrm{Span}(X_k), B) \leqslant \psi(X_k, Y)$$

because $Y \in B$. Using inequality (20), it follows

$$\psi_k(F, B) \leqslant c_{F,n} \sum_{i=1}^{\ell} \psi_{d_i}(F_i, B_i).$$ □

Now that Lemma 6.1 is proved, we can tackle the proof of Theorem 1.10.

Proof of Theorem 1.10. Let $F \in \mathfrak{I}_n(d, e)_j$. Let us show that F possesses an orthonormal family $(f_1, \ldots, f_j)$ such that for all $\ell \in \{1, \ldots, j\}$, at least $d - \ell$ coordinates of f_ℓ vanish. We proceed by induction on $\ell_0 \in \{0, \ldots, j\}$, constructing an orthonormal family $(f_1, \ldots, f_{\ell_0})$ such that for any $\ell \in \{1, \ldots, \ell_0\}$, at least $d - \ell$ coordinates of f_ℓ vanish. For $\ell_0 = 0$ the empty family has this property. If such a family $(f_1, \ldots, f_{\ell_0})$ is constructed for some $\ell_0 \in \{0, \ldots, j-1\}$, we denote by G the orthogonal complement of $\mathrm{Span}(f_1, \ldots, f_{\ell_0})$ in F; if $\ell_0 = 0$ this means $G = F$. One has $G \cap (\mathbb{R}^{n-d+\ell_0+1} \times \{0\}^{d-\ell_0-1}) \neq \{0\}$ because $\mathrm{codim}(\mathbb{R}^{n-d+\ell_0+1} \times \{0\}^{d-\ell_0-1}) = \dim G - 1$, let $f_{\ell_0+1} \in G \cap (\mathbb{R}^{n-d+\ell_0+1} \times \{0\}^{d-\ell_0-1})$ be a unitary vector. At least $d - (\ell_0 + 1)$ coordinates of this vector vanish, and it is orthogonal to $f_1, \ldots, f_{\ell_0}$. This concludes the proof by induction.

In all what follows, let $(f_1, \ldots, f_j)$ be an orthonormal family of F such that for all $\ell \in \{1, \ldots, j\}$, at least $d - \ell$ coordinates of f_ℓ vanish. Let us denote by $\underline{x}$ the vector formed with all the nonzero coordinates of the f_ℓ and denote by $N \in \{1, \ldots, jn - jd + j^2/2 + j/2\}$ its number of coordinates.

One has $\underline{x} \in \mathbb{R}^N \setminus \mathbb{Q}^N$, otherwise $(f_1, \ldots, f_j)$ would span a rational subspace of dimension j of F, which cannot be since $F \in \mathfrak{I}_n(d, e)_j$. Using Dirichlet's approximation theorem, there exist infinitely many pairs $(p, q) \in \mathbb{Z}^N \times \mathbb{N}^*$ such that $\gcd(p_1, \ldots, p_N, q) = 1$ and

$$\left\| \underline{x} - \frac{p}{q} \right\|_\infty \leqslant \frac{1}{q^{1+1/N}}. \tag{21}$$

Let us fix such a pair (p, q). For $i \in \{1, \ldots, j\}$, let us denote by r_i the subfamily of p corresponding to its coordinates approaching those of f_i, completed with zeros so that $r_i \in \mathbb{Z}^n$ is close to qf_i. For all $i \in \{1, \ldots, j\}$, one has $\| f_i - r_i/q \|_\infty \leqslant q^{-1-1/N}$.

Let $B = \mathrm{Span}(r_1, \ldots, r_j)$, and let us denote by $\pi_i^\perp(f_i)$ the orthogonal projection of f_i onto $\mathrm{Span}(r_i/q)$. One has

$$\psi\left(f_i, \frac{r_i}{q}\right) = \sin\left(\widehat{f_i, \frac{r_i}{q}}\right) = \frac{\| f_i - \pi_i^\perp(f_i) \|}{\| f_i \|} \leqslant \left\| f_i - \frac{r_i}{q} \right\| \leqslant \frac{c_1}{q^{1+1/N}} \tag{22}$$

because $\| f_i \| = 1$, with $c_1 > 0$ depending only on n. Inequality (21) gives $\| p \|_\infty - \| q\underline{x} \|_\infty \leqslant \| q\underline{x} - p \|_\infty \leqslant q^{-1/N} \leqslant 1$, so for all $i \in \{1, \ldots, j\}$: $\| r_i \|_\infty \leqslant \| p \|_\infty \leqslant 1 + \| q\underline{x} \|_\infty \leqslant c_2 q$, with $c_2 > 0$ depending only on F.

For E a subspace of $\mathbb{R}^n$ and P a family of linearly independent vectors of E, let us denote by $\mathrm{vol}_E(P)$ the volume of the parallelotope spanned by the vectors of P and considered in the Euclidean space E. Since $(r_1, \ldots, r_j)$ is a sublattice of $B \cap \mathbb{Z}^n$, one has using Theorem 2.3

$$H(B) \leqslant \mathrm{vol}_B(r_1, \ldots, r_j) \leqslant \prod_{i=1}^{j} \| r_i \| \leqslant c_3 q^j,$$

with $c_3 > 0$ depending only on F. Thus, there exists a constant $c_4 > 0$ such that

$$\frac{1}{q} \leqslant \frac{c_4}{H(B)^{1/j}}. \tag{23}$$

Let $\widetilde{F}_j = \mathrm{Span}(f_1, \ldots, f_j)$ which is a subspace of dimension j of F, and let $B_i = \mathrm{Span}(r_i)$ for $i \in \{1, \ldots, j\}$. According to Lemma 6.1 and inequality (22), one has

$$\psi_j(\widetilde{F}_j, B) = \psi_j\left(\bigoplus_{i=1}^{j} \mathrm{Span}(f_i), \bigoplus_{i=1}^{j} B_i \right) \leqslant c_5 \sum_{i=1}^{j} \psi_1(\mathrm{Span}(f_i), B_i) \leqslant \frac{c_6}{q^{(N+1)/N}}, \tag{24}$$

with $c_5, c_6 > 0$ depending only on n and F. Moreover, $F \supset \widetilde{F}_j$, so $\psi_j(F, B) \leqslant \psi_j(\widetilde{F}_j, B)$. Thus, inequalities (23) and (24) show that there exists a constant $c_7 > 0$ depending only on n and F such that

$$\psi_j(F, B) \leqslant \frac{c_7}{H(B)^{(N+1)/(jN)}} \leqslant \frac{c_7}{H(B)^{(jn-jd+j^2/2+j/2+1)/(j(jn-jd+j^2/2+j/2))}}; \tag{25}$$

hence

$$\mathring{\mu}_n(d\,|\,j)_j \geqslant \frac{jn-jd+j^2/2+j/2+1}{j^2(n-d+j/2+1/2)}$$

and the result follows from Proposition 1.4. ☐

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ELIO JOSEPH:
josephelio@gmail.com
Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, Orsay, France

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Part I